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NON-EUCLIDEAN GEOMETRY[1]

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About 2200 years ago there was published in Greek one of the most remarkable books of all times, Euclid's "Elements of Geometry". It contains a systematic exposition of the leading propositions of elementary geometry and the elementary theory of numbers. It was at once adopted by the Greeks as the standard text book on pure mathematics. The parts that relate to elementary geometry were the standard text book for centuries and are still in use in England to-day. The English school boy does not say "Geometry", he says "Euclid". On the Continent of Europe "Euclid" was superseded by Legendre's "Elements of Geometry", the first edition of which was published in 1794. A translation into English by a man named Davies was widely used in this country. (It was used at Columbia University as late as 1905). But that has been superseded by more modern American texts of which there is now a large number.

The best known English edition of Euclid is probably that of Todhunter which contains Books I to VI and parts of Books XI and XII. The most complete and the most valuable edition is "The Thirteen Books of Euclid's Elements" by T. L. Heath in three volumes with many pages of historical, critical and explanatory notes.

An edition by Robert Simson of the University of Glasgow was published in Philadelphia in 1811. The title page says that "the errors by which Theon and others have long ago vitiated these books are corrected and some of Euclid's demonstrations are restored". On the inside of the cover of a copy of this book I found written in pencil: "Q. E. D. Quid Euklid Dixit," an anonymous tribute to the authority of Euclid.

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[1] Paper read before the Twin City Mathematics Club, May 13, 1922.

Euclid's work on geometry is largely a compilation from the works of previous authors. Just how much of it was original with Euclid is unknown. Many of his theorems are known to have been due to Thales, Pythagoras, Hippocrates and others. But the material he took from other authors was rearranged by him and in some cases new proofs were substituted.

Euclid's book begins with definitions, postulates, and common notions, some of which I shall quote.

Definitions:

1. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Common Notions:

1. Things which are equal to the same thing are equal to each other.
2. The whole is greater than the part.

Postulates:

1. A straight line can be drawn from any point to any point.
2. A finite straight line may be produced continuously in a straight line.
3. All right angles are equal.
4. A circle can be described with any centre and any radius.

These common notions and postulates are often referred to as "axioms".

There is another axiom of Euclid's usually known as Postulate 5, which differs from the ones I have quoted and from all the others. It is prolix. It has different places in the different manuscripts. It is not used in the first 28 propositions and then it is used only to prove the converse of a proposition previously proved. It sounds more like a proposition than an axiom. Euclid proved propositions far more self-evident. As someone has said, he proved what every dog knows that two sides of a triangle are, taken together, greater than the third side. Yet when he had proved Proposition 28, that lines making equal angles with a transversal are parallel, in order to prove the converse he stated and used this postulate:

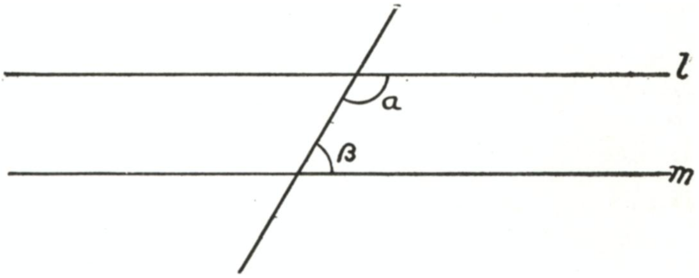
Postulate 5: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, will meet on that side on which are the angles less than two right angles".

Perhaps you are not familiar with it in this awkward form. It is equivalent to this: "Two straight lines which intersect one another can not both be parallel to the same straight line", or "Through a given point only one parallel can be drawn to a given line". This is commonly known as Playfair's form of the parallel axiom although it was known to Proclus, who wrote a commentary on Euclid's Elements in the 4th century A. D.

Euclid proved in Book I, Proposition 28 that through a point there is one parallel to a given line. Playfair's axiom states that there is only one. With this assumption Euclid's Postulate 5 is easily proved, and vice versa.

I have said that Postulate 5 sounds more like a proposition than an axiom. To make this clear, I shall quote it again: "If a straight line falling on two straight lines makes the interior angles on the same side together equal to two right angles, the two straight lines will be parallel to one another". That sounds very much like what I quoted before as Postulate 5, but as a matter of fact it is no such thing. It is Proposition 28. Euclid proved it and all the previous propositions without using Postulate 5, which I shall now read again: "If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, will meet on that side on which are the angles less than two right angles".

He used it in the proof of Proposition 29 which is as follows: "If a straight line fall on two parallel straight lines, it makes the interior angles on the same side of the line together equal to two right angles". You must keep in mind Proposition 28, Postulate 5 and Proposition 29. To this end I shall state all three very briefly in terms of a single diagram.



Prop. 28: If $\alpha + \beta = 180°$, lines l and m are parallel.
Postulate 5: If $\alpha + \beta < 180°$, lines l and m meet.
Prop. 29: If lines l and m are parallel, $\alpha + \beta = 180°$.
Remember that Euclid proved proposition 28 and then had to assume Postulate 5 in order to prove Proposition 29.

From the very beginning Postulate 5 was attacked as not self-evident and as needing demonstration, and Euclid's contemporaries and successors tried to prove it by reasoning from the other exioms. Proclus in the 4th century A. D. said: "This ought to be struck out of the postulates altogether for it is a theorem involving many difficulties which Ptolemy, in a certain book, set himself to solve and it requires for the demonstration of it a number of definitions as well as theorems. And the converse of it is actually proved by Euclid as a theorem":

The converse of it, in terms of the diagram given above is:

"If lines l and m meet, $\alpha + \beta < 180°$."

It is equivalent to Proposition 28 which is:

"If $\alpha + \beta = 180°$, lines l and m are parallel."

We have a record of noteworthy attempts to prove Postulate 5 dating from Ptolemy's attempt in the 2nd century A. D. to the present day; with others still to come although they will not be noteworthy because it has been demonstrated by mathematicians that the thing can not be done. All the attempts were failures in the sense that either they begged the question, or else they assumed something else equivalent to Postulate 5 and just as much in need of proof.

Geminus (about 50 B. C.) assumed that there exist straight lines everywhere equidistant from one another.

Ptolemy (2nd century A. D.) really begged the question.

Proclus (5th century A. D.) assumed that if a straight line intersects one of two straight lines it intersects the other also.

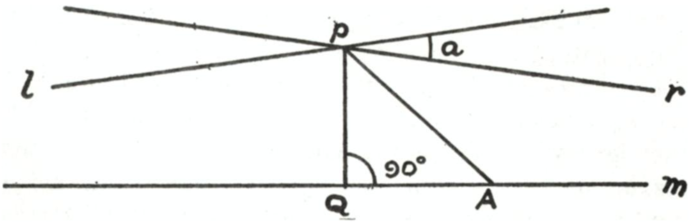
Wallis (17th century A. D.) assumed that, given any figure, there exists a similar figure similar to it of any size we please.

Legendre (1752-1833) was perhaps the greatest mathematician who, up to his time, attempted to prove Postulate 5. His different attempts appeared in the successive editions of his "Elements of Geometry" from the first in 1794 to the twelfth in 1823. His exposition brought out clearly the essential connection between the axiom of parallels and the sum of the angles of a triangle. In the first edition of his geometry the proposition that the sum of the angles of a triangle is equal to two right angles was proved analytically on the basis of the assumption that the choice of a unit of length does not affect the correctness of the proposition to be proved, which is of course equivalent to Wallis' assumption of the existence of similar figures. In the second edition he proved Postulate 5 by means of the assumption that, given three points not in a straight line, there exists a circle passing through all three. In the third edition he proved that the sum of the angles of a triangle is not greater than two right angles. But he could not prove the angle sum to be not less than two right angles. If he had succeeded in that, he would have proved the equivalent of Postluate 5, namely that the angle sum in a triangle is exactly equal to two right angles. Legendre proved this, however: "If the sum of the angles of *one* triangle is equal to two right angles, the sum of the angles of *any other* triangle is also equal to two right angles.

In 1826, a few years before Legendre's death, a Russian named Lobatschewsky published at Kasan, in Russia, a little book called "The Theory of Parallels." It was followed in 1836, 1837 by other papers on the same subject. His work was unnoticed for years. In 1840 he published a resume of it in German and in 1855, a year before he died, he published a complete exposition of it in French and Russian. As early as

1815 he was working on parallels and he made several attempts to prove Postulate 5 and some investigations resembling those of Legendre. I do not know what was in his mind when he published the booklet in 1826, but I like to think that he was trying to prove Postulate 5 by the reductio ad absurdum method. If the postulate could be proved by logical reasoning, from Euclid's other axioms, a denial of the postulate ought to lead to a contradiction and thus result in a proof by the so called indirect method. At any rate, Lobatschewsky denied that there was only one line through a point and parallel to a given line; he assumed that there was more than one. (Remember that Euclid proved the existence of one). Lobatschewsky did it in this way:

Let *m* denot a line, *A* a point on that line, and *P* a point not on that line.



Let *A* move indefinitely to the right and denote by *r* the line which is the limiting position approached by the line *PA*. In like manner let *l* be the limiting position approached by *PA* as *A* recedes indefinitely to the left. And assume that *r* and *l* are different lines. (On the Euclidean assumption, *r* and *l* are one and the same line.) The figure consisting of *P, m, r, l* is symmetrical with respect to the line *PQ* drawn from *P* perpendicular to *m*. It is evident that any line through *P* which lies within the angle (*a*) between *r* and *l* will not meet *m*. The number of such lines is infinite. So with the Euclidean definition of the word "parallel," Lobatschewsky's assumption really is that the number of parallels through *P* is not *one* but *infinite*. But Lobatschewsky used the word "parallel" with a somewhat different meaning. With reference to *P* and *m*, he called (*r*) a parallel on the right and (*l*) a parallel on the left. He classified the lines through *P*, in their relation to *m*, as "intersecting lines,'' ''non-intersecting lines,'' and ''parallels'' defined as the boundary lines between the ''intersecting'' and the ''non-intersecting'' lines.

Now if he had this substitute for the Euclidean axiom in mind as the beginning of a reductio ad absurdum proof of Postulate 5, he naturally would have tried to deduce logical consequences of his hypothesis and the Euclidean assumptions other than Postulate 5 with the expectation that sooner or later he would arrive at a *contradiction* which would be the end of the reductio ad absurdum proof. And the result would have been an indirect proof of Postulate 5. Whatever his expectation was, he never arrived at such a contradiction. He proved theorem after theorem; he pointed out that many of Euclid's theorems were just as valid with his assumption about parallels as with Euclid's assumption. He finally came to the conclusion that Euclid's Postulate 5 could not be deduced as a logical consequence of Euclid's other axioms.

If one is trying to make a reductio ad absurdum proof and fails to find the contradiction which he seeks, he is not justified in saying that no such contradiction can be found. If he has tried for 5 hours without result, he may find the contradiction in the next five minutes, and if he gives it up as a bad job, he can never be sure that he would not have succeeded if he had persevered. So the fact that Lobatschewsky did not find a contradiction did not prove that there was none to be found. It remained for later mathematicians to prove conclusively that the set of theorems developed by Lobatschewsky together with all others logically deducible from his assumptions constitute as logically consistent a geometry as the Euclidean Geometry known to you all. Cayley, an Englishman, and Klein, a German are jointly responsible for the demonstration of this fact. The details of their work are long and they involve much higher mathematics. I shall not speak of them further.

Now you may be interested in knowing some of the characteristics of this geometry of Lobatschewsky. Propositions whose proofs depend solely on the method of superposition are the same as in Euclidean Geometry. Here are some of the more important theorems that are Non-Euclidean.

1. The sum of the angles of a triangle is less than 180°. The amount by which the angle sum is less than 180° is called the "deficiency of the triangle".

2. The areas of two triangles are to each other as their deficiencies. As the area of a triangle becomes smaller, the deficiency approaches zero as its limit or, in other words, the angle sum approaches 180° as its limit and the state of affairs approaches the Euclidean. There is a finite maximum to the area of a triangle.

3. In a tri-rectangular quadrilateral, the fourth angle is acute.

4. Parallel lines continually approach one another.

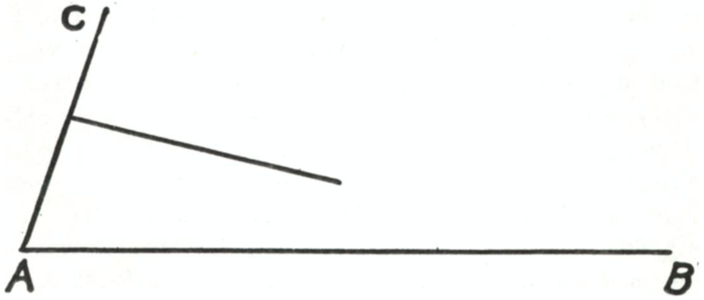
5. The angle between the line *r* and the line *PQ* in the figure given above is called the angle of parallelism for the distance *PQ*. As *PQ* increases from 0 to infinity, the angle of parallelism decreases from 90° to 0.

6. The perpendiculars erected at the middle points of the sides of a triangle are all parallel if two of them are parallel.

7. Straight lines which do not intersect and are not parallel have one and only one common perpendicular.

8. A straight line may be drawn perpendicular to a plane and parallel to any straight line not in the plane.

9. The locus of points equidistant from a straight line is not a straight line. It is called an equidistant curve.



10. Given a straight line *AB;* as its extremity *A* draw any arbitrary angle *BAC* and produce *AC* so that its perpendicular bisector shall be parallel to *AB*. The locus of *C* is called a "boundary curve" or "oricycle". *AB* is called an axis. The curve has an infinite number of such axes.

11. A boundary curve has a constant curvature. It is the limiting curve between a circle and an equidistant curve. That is, it may be thought of as a circle of infinite radius or an equidistant curve whose base line is at an infinite distance.

I have stated theorems enough to give you some idea of what this Non-Euclidean Geometry is like. I have said that its origin is due to Lobatschewsky. As a matter of fact it was developed independently by John Bolyai, a Hungarian, and it has been surmised that the same thing was done by the great German mathematician, Gauss, but he never published his results, some-say because he did not want to detract from the credit due the other two, especially his friend, Bolyai.

Euclid assumed that there was only one straight line through a point and not meeting a given line. Lobatschewsky assumed that the number was greater than one and therefore infinite. You may ask why did not some one assume that there was no such line and thus develop a different Non-Euclidean Geometry. The answer is, someone did. His name is Rieman. He denied the Euclidean Postulate 5 by saying that every two straight lines meet in one and only one point. What then of Euclid's Proposition 28 which proves the existence of one straight line through a given point and parallel to a given straight line. The proof of that proposition depended on Euclid's assumption that a straight line is of infinite length. Rieman denied that too. He assumed a straight line to be ''unbounded'' but not ''infinite''. That there is a real distinction between the two, I think you will see if you think of the surface of a sphere. It is unbounded but not infinite. So the proof of Euclid's Proposition 28 fails in Rieman's Non-Euclidean Geometry. Here are some of the more important propositions of it. (Of course many theorems are the same as in Euclidean and in Lobatschewsky's Non-Euclidean Geometry.)

1. The sum of the angles of a triangle is greater than 180°. The amount by which the angle sum is greater than 180° is called the ''excess'' of the triangle.

2. The areas of two triangles are to each other as their ''excesses''. As the area of a triangle becomes smaller, the excess

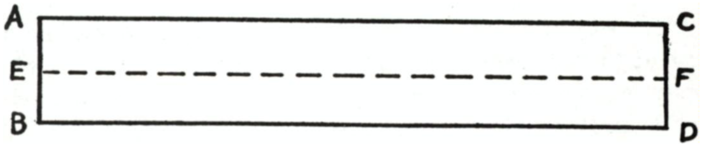
approaches zero as its limit or, in other words, the angle sum approaches 180° as its limit and the state of affairs approaches the Euclidean.

3. In a tri-rectangular quadrilateral, the fourth angle is obtuse.

4. A straight line is of finite length. It returns unto itself.

5. All perpendiculars to a given straight line meet in a point called its pole which is a half line-length away from the given line. The line is called the polar of the point.

6. A straight line does not divide the plane into two pieces. That is a line can be drawn in a plane from one side of a straight line to another without crossing the line. The plane is what is called a unilateral surface. To get an idea of what a unilateral surface is, take a strip of paper *ABCD*



and join its two ends so that *C* coincides with *B* and *D* coincides with *A*. Then cut along the line *EF* and see if the cut separates the paper into two parts.

7. The locus of points at a given distance from a straight line is a circle having its center at the pole of the line.

8. There is a trirectangular triangle.

9. A straight line is the limit approached by a circle where the length of the radius approaches one half of a line-length.

10. The plane is unbounded but not infinite.

11. All planes perpendicular to a straight line meet in a straight line called the conjugate of the given line. The relation between the two lines is reciprocal and all the points of one are a half line-length away from the other line.

12. There are straight lines not in the same plane which have an infinite number of common perpendiculars and are everywhere equidistant.

13. There are no similar figures in this Non-Euclidean Geometry.

Now you know a little something about two Non-Euclidean Geometries. I am going to suggest a third. It resembles the one I have just talked about and it has these characteristics.

1. The straight line is of finite length and returns unto itself. It is the shortest line joining two points.
2. Every two straight lines meet in two points.
3. The sum of the angles of a triangle is greater than 180°. The amount by which the angle sum is greater than 180° is called the "excess" of the triangle.
4. The areas of two triangles are to each other as their "excesses". As the area of a triangle becomes smaller, the excess approaches zero as its limit.
5. In a tri-rectangular quadrilateral the fourth angles is obtuse.
6. All perpendiculars to a given straight line meet in two points each a fourth of a line-length away from the given line. They are called the poles of the line. The line is called the polar of each of the points.
7. The locus of points at a given distance from a straight line is a circle having its centre at one of the poles of the line.
8. There is a tri-rectangular triangle.
9. A straight line is the limit approached by a circle as the length of the radius approaches one fourth of a line-length.
10. There are no similar figures in this Non-Euclidean Geometry.

You have noticed that this Non-Euclidean Geometry resembles the Rieman Non-Euclidean Geometry, but that there are important differences, namely these. In this geometry last mentioned, two straight lines meet in two points, not one; the distance from a straight line to its polar is one fourth and not one-half of a line-length; a straight line *does* divide the plane into two parts; the plane is *not* unilateral. In other respects it resembles the Rieman Non-Euclidean Geometry. But in a very real sense it is Euclidean Geometry, and it is well known to all of you. If you will substitute "sphere" for "plane" and "great circle" for "straight line" and "spherical excess" for "excess," it will become what is known as spherical geometry, that is, the geometry of great circles on the surface of a sphere. If you have difficulty in thinking of great circles on a sphere as analogus to straight lines in a plane, remember that the great circle is the shortest line on a sphere connecting two points of the sphere, and the great circle is the line traced out by a point on the sphere which starts in a given direction and turns neither to the right nor to the left. And if geometry had been first developed for purposes of navigation on the great oceans, spherical geometry might have been developed before Euclidean plane geometry, just as spherical trigonometry was developed before plane trigonometry.

Perhaps you are now inclined to ask: "What is the significance of these Non-Euclidean Geometries and what is their relation to the space in which we live?" The easy answer to give is that Non-Euclidean Geometry is a branch of pure mathematics developed by mathematicians to satisfy their intellectual curiosity and that it has no relation to the space in which we live. But that is not the only answer that has been given to the question.

Geometry used to be thought of by almost everybody as an exact science having absolute certainty, as a science developed by logical reasoning from a set of unalterable axioms which were self-evident and a part of our intuition of space. The idealists of the 17th and 18th centuries, who held that certain knowledge, independent of experience, was possible about the real world, had only to point to geometry as an illustration. They said that no one but a madman would doubt its validity and no one but a fool would deny its objective reference.

On the other hand the empiricists of that time asserted that geometry had no certainty of a different kind from that of mechanics. They said that only the perpetual presence of spatial impressions made our experience of the truth of the axioms of geometry so wide as to seem absolute certainty. In saying this they went very much against the common sense of their day.

These two opposite views were held before the development of Non-Euclidean Geometry and they are still held by mathematicians and philosophers and by those who claim to be neither. If you join either group, you will have distinguished company.

Those who hold that the Euclidean axioms of geometry are intuitional dispose of Non-Euclidean Geometry by saying that the logical possibility of Non-Euclidean systems is irrelevant because the basis of geometry is not logic but intuition.

Those in the other group, who do not regard Euclidean Geometry as having absolute certainty, say that space as a whole may be Euclidean or it may be Non-Euclidean, and there is no means of determining which. You may say: "Why not measure the sum of the angles of a triangle and thus determine whether space is Euclidean or not?" Well, the measurement of the angles of the best approximation we can make to a geometric triangle will show an angle sum of approximately 180°. If the measurement does not show exactly 180°, the difference may well be attributed to the imperfection of measuring instruments. Therefore, the part of space in which the triangle lies is at least approximately Euclidean, and the use of Euclidean Geometry for practical purposes on this earth is justified. But the only triangles available for measurement are small as compared with the whole of space and it is characteristic of the two Non-Euclidean Geometries we have described that the angle sum is approximate 180° when the triangle is small.

So the question as to whether space is Euclidean or Non-Euclidean can not be determined by measuring the sum of the angles of a triangle. If space is Non-Euclidean, the angle sum of a triangle will not be 180°, but the difference may be too small to be detected by any means at our disposal. It has been suggested that a measurement be made of the angle sum of a triangle whose vertices are three fixed stars. Such a triangle ought to be large enough to furnish a definite result. But this measurement can not be made except by assuming that a ray of light is a straight line. But it is well known that a ray of light is not straight unless the medium through which it passes is homogeneous. So the result of the measurement, whatever it indicated, would be questioned and attributed to the bending of light rays by refraction.

Poincare, who was rated as one of the very greatest mathematicians in the world when he died a few years ago, maintained that the question whether Euclidean or Non-Euclidean Geometry should be accepted was a matter of convenience and convention and not a matter of truth. He said that the axioms were definitions in disguise and that the choice between them was arbitrary.

A little while ago, I spoke of measuring the angles of the *best approximation we can make to a geometric triangle.* I had in mind the fact that there are no such things in physical nature as geometric points, lines and planes. The geometric entities of which we talk so freely are, I suppose, abstractions of things we do see in nature. There are very small particles of matter which approximate what we call a geometric point; there are very fine threads, which may be stretched so that we call them straight, which approximate geometric straight lines, or rather segments of geometric straight lines. The Geometric straight line is infinite. And there are flat smooth table tops whose thin top layers of varnish approximate pieces of geometric planes. In other words, these geometric points, lines, and planes are creations of the intellect with only approximate counterparts in nature. When one thinks of this fact, Poincare's remarks quoted above seem quite sensible, and one is inclined to replace the question ''Which is true, Euclidean or Non-Euclidean Geometry?'' by the question ''Which is the more convenient, Euclidean or Non-Euclidean Geometry?'' There is no doubt about the answer to this latter question. Euclidean Geometry is much more convenient than either the Lobatschewsky or the Rieman Non-Euclidean Geometry.

What then is the practical value of Non-Euclidean Geometry? Well, many a branch of mathematical science has been developed by mathematicians to satisfy intellectual curiosity with no idea that it would have any practical value. And later it has been found to have great practical value. This has happened so many times that it would be foolish for anyone to claim that a branch of mathematics which has no practical value today will never have any practical value.

In the Scientific American for the year 1920 there is a short article entitled ''That parallel postulate'' and there are other articles in that magazine for 1920 and 1921 about Non-Euclidean Geometry, especially that of Lobatschewsky. In them you will find some of the things I have been talking about. Why do you suppose they are there? You may remember that, about that time, the newspapers gave much comment, but little light, on "Einstein's Theory of Relativity." The Scientific American offered a prize of $5000 for the best short essay on the subject of "Relativity" and the brief articles on Non-Euclidean geometry were put in to prepare the readers of the magazine for such statements as this, which were to appear later in the Scientific American, that the general theory of relativity leads to the conclusion that space may be finite though unbounded and therefore Non-Euclidean. As a matter of fact Non-Euclidean Geometry plays an important part in the literature of Einstein's "Theory of Relativity" as does also the geometry of four dimensions which to most people has seemed less practical and more weird and non-sensical than Lobatschewsky's Non-Euclidean Geometry. The pure mathematicians found a great deal of satisfaction in watching the physicists rush to learn what the mathematicians had done with Non-Euclidean Geometry when they, the physicists, discovered that some knowledge of that subject was necessary for the understanding of Einstein's work.

Having mentioned four dimensional space, I think it is time for me to stop before I am tempted to speak at length on that fascinating subject, although space of more dimensions than three is certainly "Non-Euclidean" in a sense and therefore what I might say on the subject would be appropriate to the title of this paper.

I close with a quotation from David Hilbert one of the greatest of modern mathematicians: "The most suggestive and notable achievement of the last century is the discovery of Non-Euclidean Geometry."